

σ -Model on de Sitter Space

Peter C. Aichelburg, Christiane Lechner

Institut für Theoretische Physik, Universität Wien,
Boltzmanngasse 5, A - 1090 Wien, Austria e-mail:
pcaich@doppler.thp.univie.ac.at, lechner@galileo.thp.univie.ac.at

Abstract

We discuss spherically symmetric, static solutions to the $SU(2)$ σ -model on a de Sitter background. Despite of its simplicity this model reflects many of the features exhibited by systems of non-linear matter coupled to gravity e.g. there exists a countable set of regular solutions with finite energy; all of the solutions show linear instability with the number of unstable modes increasing with energy.

1 Introduction

With the unexpected discovery of spherically symmetric static solutions to the Einstein Yang Mills system by Bartnik and McKinnon [1] new interest focused on particle-like solutions for non-linear matter coupled to gravity. Many of these systems under consideration have been shown to allow for asymptotically flat soliton and black hole solutions (for a recent review on the EYM-system see [2]). More recently Brodbeck et. al. [3] investigated the EYM-equations including a cosmological constant and showed, that a rich spectrum of solutions exists depending on the dimensionless parameter in the theory.

An extremely simple model that lives on a compact 3-space was previously considered by Bizon [4]: he looked at harmonic maps from (\mathbb{S}^3, G) to (\mathbb{S}^3, G) , where G denotes the metric of constant curvature on \mathbb{S}^3 . In the present paper we modify his model to harmonic maps from de Sitter spacetime to \mathbb{S}^3 , i.e. we consider a $SU(2)$ σ -model that lives on a de Sitter spacetime with positive cosmological constant. This model has the advantage of being more physical while keeping the simplicity. We look for spherically symmetric static solutions in a static coordinate patch of this geometry. Although the slices of constant time are \mathbb{S}^3 , the field equations resulting from this model differ from those considered by Bizon. Numerical analysis of these equations reveals the existence of a discrete sequence of static solutions with finite energy. The energy of this sequence is bounded from above by a limiting (singular) solution. We show that all of these solutions are unstable under linear (time dependent) perturbations, the number of unstable modes increasing with energy. Some care is taken to analyze the high excitations. To check the eigenvalues of the unstable modes we compare them to the perturbations of the limiting solution. For this we use a self-adjoint extension of the operator governing these perturbations.

2 The Non-linear σ -Model

We take (M, g) to be de Sitter space (with positive cosmological constant Λ). The non-linear σ -model under consideration is a harmonic map $X : (M, g) \longrightarrow (\mathbb{S}^3, G)$, defined by the stationary points of the action $S_{\mathcal{M}} = \int d^4x \mathcal{L}_{\mathcal{M}}$ with the Lagrange density

$$\mathcal{L}_{\mathcal{M}} = -\sqrt{-g} \frac{f_{\pi}^2}{2} g^{\mu\nu} \nabla_{\mu} X^A \nabla_{\nu} X^B G_{AB}(X). \quad (1)$$

This yields the field equations

$$g^{\mu\nu} (\nabla_{\mu} \nabla_{\nu} X^A + \tilde{\Gamma}_{BC}^A \nabla_{\mu} X^B \nabla_{\nu} X^C) = 0, \quad (2)$$

where $\tilde{\Gamma}_{BC}^A$ denote the Christoffel symbols with respect to the metric G_{AB} of the target manifold. We choose coordinates $(X^A) = (f, \Theta, \Phi)$, so that G assumes the form $G_{AB} = \text{diag}(1, \sin^2 f, \sin^2 f \sin^2 \Theta)$. It is well known that part of the de Sitter

space can be covered by a static frame i.e. coordinates $(t, r, \vartheta, \varphi)$ in which the metric becomes

$$ds^2 = - \left(1 - \frac{r^2}{\alpha^2}\right) dt^2 + \left(1 - \frac{r^2}{\alpha^2}\right)^{-1} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2, \quad (3)$$

with $0 \leq r < \alpha$ and $\alpha^2 = 3/\Lambda$, Λ being the cosmological constant. The singularity at $r = \alpha$ signals the breakdown of the static coordinates. We seek for regular solutions of Eqs. (2) in the domain, where the metric takes the static form (3) and require the mapping to be static, i.e. $\partial_t X^A = 0$, and spherically symmetric. The latter is guaranteed by the Hedgehog-Ansatz:

$$f(x^\mu) = f(r), \Theta(x^\mu) = \vartheta, \Phi(x^\mu) = \varphi. \quad (4)$$

In order to map the pole $r = 0$ to a single point of \mathbb{S}^3 , we require the function $f(r)$ to obey the boundary condition $f(0) = 0$. Inserting this Ansatz into Eqs.(2) yields the non-linear ordinary differential equation

$$\left(r^2 \left(1 - \frac{r^2}{\alpha^2}\right) f'\right)' - \sin 2f = 0. \quad (5)$$

The energy associated with an excitation of the form (4) is given by

$$E = 4\pi f_\pi^2 \int_0^\alpha r^2 dr \left\{ \left(1 - \frac{r^2}{\alpha^2}\right) f'^2 + \frac{2}{r^2} \sin^2 f \right\} \quad (6)$$

where a factor of 2 has been included, because the r -integration runs twice from 0 to α .

Note, that the energy functional $E[f]$ (6) may be interpreted again as a spherically symmetric harmonic map, this time from $(\mathbb{S}^3, (1 - \frac{r^2}{\alpha^2})^2 G)$ to (\mathbb{S}^3, G) .

To our knowledge the only analytic solutions to (5) are the trivial solution $f \equiv 0 \pmod{\pi}$ with energy $E_0 = 0$ and the solution $f \equiv \pi/2$, which will be called singular, since it fails to obey the boundary condition at $r = 0$ (see section 6).

3 Numerical Results

In the following we introduce a new independent variable x defined by $r = \alpha \sin \chi$ and $x = -\ln \tan \chi/2$. The coordinate x tends from $x = 0$ ($r = \alpha$) to $x = \infty$ ($r = 0$). We also define a new dependent variable h by $h = f - \pi/2$. In these new variables equation (5) becomes

$$h'' - \left(\tanh x - \frac{1}{\sinh x \cosh x} \right) h' + \sin 2h = 0, \quad (7)$$

where prime now denotes differentiation with respect to x .

Equation (7) is similar to the one studied by P.Bizon [4]. Ours differs from his by the term $(\sinh x \cosh x)^{-1}$ which causes a singular point of the differential equation at $x = 0$ (corresponding to the breakdown of the static frame at $r = \alpha$). For the solutions to be regular at $x = 0$, $h'(0)$ has to vanish. The other singular point is at $x = \infty$ where $h(\infty) = \pm\pi/2 \pmod{\pi}$. The formal power series expansions of $h(x)$ around these singular points give:

$$\begin{aligned} h(x) &= b - 1/4(\sin 2b)x^2 + O(x^4) & \text{for } x \simeq 0 & \quad \text{and} \\ \pm h(x) &= \pi/2 - 2ce^{-x} + O(e^{-3x}) & \text{for } x \rightarrow \infty. \end{aligned} \quad (8)$$

One can show that near $x = 0$ as well as $x = \infty$, there exists a family of solutions, each of them depending only on a single parameter, $b \equiv h(0)$ and c respectively. Since with $h(x)$ also $\tilde{h}(x) = n\pi \pm h(x)$ is solution to equation (7), we restrict the parameter b to the range $0 < b < \pi/2$.

We integrated equation (7) numerically, starting at $x = 0$, trying to find values for b , such that $h(x)$ shows the correct asymptotic behavior for large x . With the values found for b we used the two-point shooting and matching method to determine the parameter c . Both numerical routines, d02cbf and d02agf, were provided by the NAG-Library [6]. The numerical analysis suggests the existence of a countable family of globally regular solutions of Eq.(7) which may be labeled by their number of zeros, and indeed a rigorous proof of existence can be given along the lines in [4].

Figure 1 shows the first four solutions $h_n(x)$, where the sign of b_n has been chosen such that $h_n(\infty) = -\pi/2$. The numerical results b_n and c_n for the first four solutions are displayed in Table 1.

n	b_n	c_n	b_n^2/b_{n+1}^2	c_{n+1}/c_n	$E_n/4\pi f_\pi^2 \alpha$
1	0.4458536	5.0270678	10.636094	10.831606	1.8304231
2	0.1367103	54.451223	10.734453	10.747914	1.9842302
3	0.0417264	585.23706	10.747677	10.748889	1.9985328
4	0.0127278	6290.6484	10.748955	10.749068	1.9998635
' ∞ '			10.749087	10.749087	2

Table 1: Numerical values of the parameters b_n , c_n , the ratios $b_n^2/b_{n+1}^2, c_{n+1}/c_n$ and the energy E_n of the first four excitations.

4 Large Excitations

In what follows we give arguments for understanding the numerical value of the ratios formed from the integration constants for consecutive values of n as listed in Table 1. The line of reasoning, which is based on approximating Eq. (7) for large

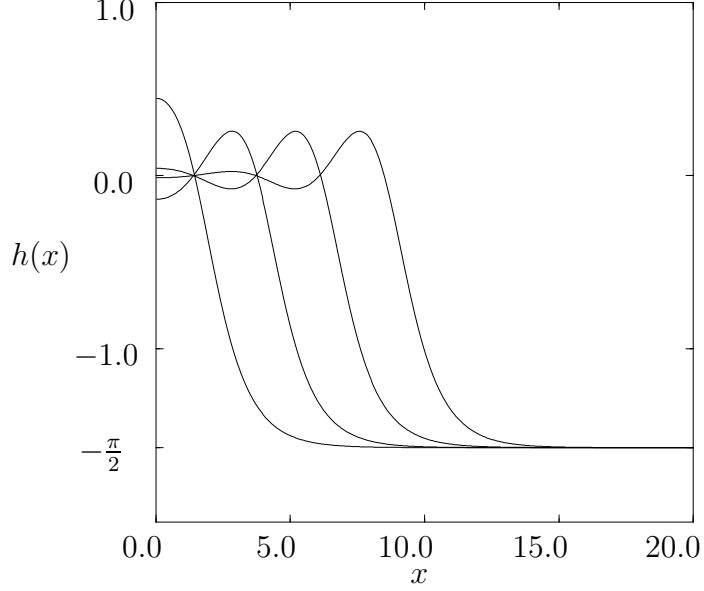


Figure 1: The first four solutions $h_n(x)$ (The sign of b_n has been chosen such that $h_n(\infty) = -\frac{\pi}{2}$).

excitations by linear respectively translation invariant differential equations, is the same as in [4], but for completeness we will repeat it here.

For large n one can distinguish essentially three regions for equation (7):

Region I, for $0 \leq x \leq 4$, where numerical analysis tells us, that h is small, so that Eq. (7) is well approximated by

$$h'' - \left(\tanh x - \frac{1}{\sinh x \cosh x} \right) h' + 2h = 0. \quad (9)$$

Since equation (9) is linear $h_n(x) \simeq b_n g(x)$ for large n , where $g(x)$ solves equation (9) with initial conditions $g(0) = 1$ and $g'(0) = 0$.

In *region II*, $4 \leq x \leq 2n$, where h is still small, one can approximate equation (7) for large n by

$$h'' - h' + 2h = 0, \quad (10)$$

which is linear as well as translation invariant. Eq. (10) is solved by $h_n \simeq b_n g(x)$ with $g(x) \propto e^{x/2} \sin(\sqrt{7}x/2 + \delta)$.

Finally in *region III*, for $x \geq 2n$, equation (7) becomes

$$h'' - h' + \sin 2h = 0, \quad (11)$$

and is solved asymptotically by $\pm h_n(x) = \pi/2 - 2c_n e^{-x}$.

Now, by using translation invariance of Eqs. (10) and (11) as well as the asymptotic behavior of h_n , one obtains in region II and III for two consecutive solutions:

$$h_{n+1}(x) \simeq h_n(x - \ln c_{n+1}/c_n), \quad (12)$$

if we choose h_{n+1} and h_n to have the same asymptotic behavior. (which is possible by choice of sign). In the regions I and II, due to the linearity of Eqs. (9) and (10) and $h_n(0) = 0$, h_{n+1} and h_n are connected by

$$h_{n+1} = \frac{b_{n+1}}{b_n} h_n. \quad (13)$$

Using the solution to Eq. (10) and the fact, that, in region II, h_{n+1} has one zero more than h_n , one determines the parameter values of consecutive solutions to behave like

$$b_n^2/b_{n+1}^2 \simeq c_{n+1}/c_n \simeq e^{2\pi/\sqrt{7}} \quad (14)$$

for large n .

5 Stability Analysis

We now discuss linear stability of the solutions h_n under time dependent spherically symmetric perturbations of U , i.e. we use the Ansatz:

$$h(r, t) = h_n(r) + \delta h(r, t) \quad (15)$$

where $h_n(r)$ is solution to the static field equation (7) and $\delta h(r, t)$ is considered to be small, so that we retain only terms linear in δh . The full time dependent field equation is:

$$\left(r^2 \left(1 - \frac{r^2}{\alpha^2} \right) h' \right)' + \sin 2h = \frac{r^2}{\left(1 - \frac{r^2}{\alpha^2} \right)} \ddot{h}. \quad (16)$$

Inserting (15) into (16) and linearizing in δh gives:

$$\left(r^2 \left(1 - \frac{r^2}{\alpha^2} \right) \delta h' \right)' + 2 \cos(2h_n) \delta h = \frac{r^2}{\left(1 - \frac{r^2}{\alpha^2} \right)} \delta \ddot{h}. \quad (17)$$

We set

$$\delta h(r, t) = \xi(r) e^{-i\sigma t}, \quad (18)$$

which leaves us with

$$\left(r^2 \left(1 - \frac{r^2}{\alpha^2} \right) \xi' \right)' + 2 \cos(2h_n) \xi = \frac{-\sigma^2 r^2}{\left(1 - \frac{r^2}{\alpha^2} \right)} \xi. \quad (19)$$

If this eigenvalue problem has any modes with $\sigma^2 < 0$, δh would grow exponentially in time and our solutions would be linearly unstable.

The energy of the time-dependent field is given by

$$E = 4\pi f_\pi^2 \int_0^\alpha r^2 dr \left(\frac{r^2}{\left(1 - \frac{r^2}{\alpha^2}\right)} \dot{h}^2 + \left(1 - \frac{r^2}{\alpha^2}\right) h'^2 + \frac{2}{r^2} \cos^2 h \right)$$

Regular perturbations have to satisfy the boundary conditions $\xi(\alpha) = 0$ and $\xi(0) = 0$. Using the coordinate x Eq. (19) reads

$$\sinh^2(x)\xi'' + \tanh(x)(1 - \sinh^2(x))\xi' + 2\cos(2h_n)\sinh^2(x)\xi = -\lambda\xi, \quad (20)$$

where $\lambda = \sigma^2\alpha^2$.

The asymptotic behavior of ξ is:

$$\begin{aligned} \xi &\sim x^{\sqrt{-\lambda}} && \text{for } x \sim 0 \text{ and} \\ \xi &\sim e^{-x} && \text{for } x \rightarrow \infty. \end{aligned} \quad (21)$$

Using routine d02kdf of the Fortran NAG-Library [6] we find numerically (up to $n = 3$) that the n -th "excitation" has n negative eigenvalues, i.e. n unstable modes. The numerical values of the first three negative eigenvalues of Eq. (20) for the first six excitations are displayed in Table 2.

n	λ_1^n	λ_2^n	λ_3^n
1	-2.111967		
2	-2.065756	-183.16864	
3	-2.061672	-180.59393	-21007.87
4	-2.061294	-180.3407	-20715.26
5	-2.061270	-180.3169	-20686.48
6	-2.061256	-180.3148	-20683.78

Table 2: Numerical values of the first three negative eigenvalues

6 The Limiting Solution

From Eq.(14) we conclude that the amplitudes in region I and II decrease with n as $e^{-n\pi/\sqrt{7}}$. At the same time the region of oscillations extends to larger and larger values of x . Thus for $n \rightarrow \infty$ the solution $h_n(x)$ tends point wise to zero for any finite x .

The limiting solution $h_\infty(x) = 0$ does not satisfy the required boundary conditions at $x = \infty$, and as a consequence its energy density is singular. Nevertheless the

total energy (6) is finite, $E_\infty = 8\pi f_\pi^2 \alpha$, which can be shown to be the upper bound for the energy of all regular solutions h_n :

Expressing the static Eq. (5) in terms of h , multiplying by h leads after a partial integration to the relation

$$\int_0^\alpha dr \, r^2 \left(1 - \frac{r^2}{\alpha^2}\right) h'^2 = \int_0^\alpha dr \, h \sin 2h. \quad (22)$$

Thus for regular solutions the energy functional $E[h]$ can be written as

$$E[h] = 4\pi f_\pi^2 \int_0^\alpha dr (h \sin 2h + 2 \cos^2 h). \quad (23)$$

Since the regular solutions satisfy $|h| \leq \pi/2$, the integrand of (23) has a maximum for $h = 0$.

The reason for our interest in the singular solution $h(x) = 0$ is, that the eigenvalues of the linear perturbations about regular static solutions seem to converge to some value λ_k^∞ for fixed k as $n \rightarrow \infty$. (See Table 2). So the question is, can we find the limiting values by calculating perturbations of the singular solution? This would yield a check of the numerics and moreover tell us, that we really have obtained all eigenvalues in the stability analysis.

Let us first have a look at Eq. (20) in the limit $n \rightarrow \infty$, i.e. $h_\infty \equiv 0$. Introducing a new variable $w = -\sinh^2 x$ and setting $\xi(w) = (-w)^{\sqrt{|\lambda|}/2} y(w)$ leaves us with a hyper-geometric differential equation

$$\begin{aligned} w(1-w)y''(w) + \left(\sqrt{|\lambda|} + 1 - w(\sqrt{|\lambda|} + \frac{1}{2})\right)y'(w) + \\ + \left(\frac{\sqrt{|\lambda|}}{4} - \frac{|\lambda|}{4} - \frac{1}{2}\right)y(w) = 0, \end{aligned} \quad (24)$$

which is solved by the hyper-geometric Function $F(a, b, c; w) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{n! (c)_n} w^n$, where $(a)_n = a(a+1) \cdots (a+n-1)$; $(a)_0 = 1$ etc. and

$$a = -1/4 + \sqrt{|\lambda|}/2 + i\sqrt{7}/4,$$

$$b = -1/4 + \sqrt{|\lambda|}/2 - i\sqrt{7}/4,$$

$$c = 1 + \sqrt{|\lambda|}.$$

Analytic extension of $F(a, b, c; w)$ for $|w| > 1$ gives

$$\begin{aligned} \xi(w) = & (-w)^{1/4 - i\sqrt{7}/4} P(\lambda) F(a, 1 - c + a, 1 - b + a; 1/w) + \\ & + (-w)^{1/4 + i\sqrt{7}/4} \bar{P}(\lambda) F(b, 1 - c + b, 1 - a + b; 1/w), \end{aligned} \quad (25)$$

where $\bar{}$ denotes complex conjugation and

$$P(\lambda) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}.$$

In order to get a spectrum of eigenvalues, that may serve as a limit to the spectra of regular perturbations, we need to specify boundary conditions at $w = -\infty$ ($r = 0$). This is not straightforward, since the solution $h_\infty \equiv 0$ is already singular. In what follows we make use of an idea which is due to P. Bizon [5]:

We first note, that the differential operator appearing in equation (20), rewritten in the coordinate $w = -\sinh^2 x$,

$$\tau_n := 4w(1-w)^{3/2} \frac{d}{dw} \left(\frac{w}{(1-w)^{1/2}} \frac{d}{dw} \right) - 2w \cos(2h_n) \quad (26)$$

is formally self-adjoint with respect to the scalar product

$$\langle \xi, \eta \rangle = \int_{-\infty}^0 \frac{dw}{(-w)(1-w)^{3/2}} \bar{\xi} \eta, \quad (27)$$

i.e.

$$\langle \tau_n \xi, \eta \rangle = \langle \xi, \tau_n \eta \rangle + \left(\frac{4(-w)}{(1-w)^{1/2}} ((\bar{\xi})' \eta - \eta' \bar{\xi}) \right) \Big|_{-\infty}^0. \quad (28)$$

The boundary terms vanish for (regular) perturbations of regular solutions h_n . It is this property that we would like to impose on the singular perturbations. For solutions of the eigenvalue problem (19) this requirement yields the condition, that the ratio of coefficients in (25) $\frac{\bar{P}}{P} = e^{-2i\arg(P)}$ has to be independent of λ . In fact, admitting solutions that behave for $w \rightarrow -\infty$ as in (25) with $\frac{\bar{P}}{P}$ fixed, corresponds to a self-adjoint extension of the operator τ_∞ , (see Appendix).

We are now free to choose one of the eigenvalues, say $\lambda_1^\infty = -2.06126$. The spectrum of τ_∞ then is determined by $e^{2i\arg(P(\lambda_k))} = e^{2i\arg(P(\lambda_1))}$, or equivalently

$$\arg(P(\lambda_k)) = \arg(P(\lambda_1)) + (k-1)\pi. \quad (29)$$

We obtain e.g. $\lambda_2^\infty = -180.315$ and $\lambda_3^\infty = -20683$ in good agreement with Table 2.

7 Outlook

In this paper we have analyzed spherically symmetric solutions to the non linear sigma model on de Sitter space with a positive cosmological constant. These solutions are regular everywhere in a static coordinate patch of this spacetime i.e. up to the cosmological horizon. Despite the simplicity of the model it shares many of the properties of matter models coupled to gravity and may therefore help to understand

why such excitations exist. Furthermore we expect that when the back reaction is taken into account, at least for weak coupling, these solutions will persist.

The authors would like to thank P. Bizon, S. Husa and H. Narnhofer for helpful discussions, moreover we wish to thank S. Husa for his support regarding the numerical computations.

A Appendix

The perturbation equations (19) can be transformed into a one-dimensional Schrödinger equation by introducing new dependent and independent variables:

$$\xi = \frac{\phi}{r} \quad \text{and} \quad \rho = \frac{1}{2} \ln \frac{\alpha + r}{\alpha - r} \quad (30)$$

Eqs. (19) then become

$$-\frac{d^2}{d\rho^2}\phi - 2\frac{(\cos(2h_n) + \tanh^2 \rho)}{\sinh^2 \rho}\phi = \lambda\phi. \quad (31)$$

For small ρ the potential behaves as $\frac{2}{\rho^2}$ for regular perturbations and as $-\frac{2}{\rho^2}$ for the singular case $h_\infty \equiv 0$.

In order to determine domains (if possible) on which the operators are self-adjoint, it is sufficient to consider the operators $\tau_0 = -\frac{d^2}{d\rho^2} + \frac{2}{\rho^2}$ for the regular case, $\tau_{0,\infty} = -\frac{d^2}{d\rho^2} - \frac{2}{\rho^2}$ for the singular case respectively. These operators have been studied by H. Narnhofer [7]:

It turns out that the operator τ_0 is self-adjoint on the domain

$$\mathcal{D}(\tau_0) = \{\phi, \phi' \text{ abs. cont.}, \phi'' \in \mathcal{H}, \phi(0) = \phi'(0) = 0\}, \quad (32)$$

where $\mathcal{H} = L^2(d\rho, (0, \infty))$.

$\tau_{0,\infty}$ is hermitian but not self-adjoint on the above domain, since $\dim(\text{Ker}(\tau_{0,\infty}^* \mp i)) = n_\pm = 1$. Solutions of $(\tau_{0,\infty}^* \mp i)\phi_\pm = 0$ are $\phi_+ \propto \rho^{1/2}(J_{i\sqrt{7}/2}(\rho e^{i\pi/4}) - e^{-\sqrt{7}\pi/2} J_{-i\sqrt{7}/2}(\rho e^{i\pi/4}))$, $\phi_- \propto \bar{\phi}_+$ respectively.

By adding linear combinations of ϕ_\pm of the form

$$\phi_+ + e^{2i\gamma}\phi_- \quad (33)$$

to the domain $\mathcal{D}(\tau_0)$ one obtains a one parameter family of self-adjoint extensions τ_γ of the operator $\tau_{0,\infty}$. Solutions (25) of the eigenvalue equation belong to $\mathcal{D}(\tau_\gamma)$, if one chooses $\frac{\bar{P}}{P} = 2^{-i\sqrt{7}/2} \Gamma(1 - i\sqrt{7}/2) \sin(\gamma - \frac{i\sqrt{7}\pi}{8}) / \Gamma(1 + i\sqrt{7}/2) \sin(\gamma + \frac{i\sqrt{7}\pi}{8})$.

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